On integrals of Quantum Supergroup $U_q gl(n|m)$

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Abstract

A linear basis of quantum supergroup $U_q gl(1|1)$ is given, and the nonexistence of non-zero right integral and non-zero left integral on $U_q gl(1|1)$ are proved. For some Hopf subalgebras of $U_q gl(1|1)$, we construct non-zero right integrals and non-zero left integrals on them.

1 Introduction

Integrals of Hopf algebras, in particular, integrals of quasitriangular ribbon Hopf algebras play an important role in construction of Hennings type of invariants of 3-manifolds [1]. Without using representation theory of Hopf algebras, Hennings points out a method to obtain 3-manifold invariants. According to Hennings, once directly labeling the link diagrams by elements of

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quasitriangular ribbon Hopf algebras, a universal invariants of links can be obtained. In order to get 3-manifold invariants, certain linear functionals of the Hopf algebra must be constants for a series of algebraic representations of Kirby moves over a family of links that can be transformed mutually under Kirby moves. The left integrals provide candidates for those linear functionals. Over the past years, researchers have been looking for concrete examples of quasitriangular ribbon Hopf algebras to try to work out invariants for simple 3-manifolds [2][3][4]. As we know, all examples in the field are non-graded quasitriangular ribbon Hopf algebras. In this paper, we consider integrals of quantum supergroups, specifically, the simplest Z_2-graded quantum group $U_q gl(1|1)$. Although there is no non-zero right (or left) integrals on $U_q gl(1|1)$, for some interesting subalgebras of $U_q gl(1|1)$, we get non-zero right integrals and left integrals on them.

The paper is organized as follows. In Section 2 we recall basic structure of $U_q gl(1|1)$ and give a linear base of $U_q gl(1|1)$. In Section 3 we discuss integrals on $U_q gl(1|1)$, and we prove that there is no non-zero right integral or non-zero left integral on $U_q gl(1|1)$. Section 4 gives several Hopf subalgebras of $U_q gl(1|1)$, and we construct right integrals and left integrals for them.

2 Quantum supergroup $U_q gl(1|1)$

2.1 Superalgebra $gl(1|1)$

Recall that a vector superspace of super-dimension $(p|q)$ is a Z_2-graded vector space $V = V_0 \oplus V_1$ whose 0-summand $V = V_0$ has dimension p and 1-summand $V = V_1$ has dimension q [7]. A complex vector superspace $C^p \oplus C^q$ is denoted by $C^{p|q}$. The 0-summand of the superspace is called *bosonic part* and vectors belonging to *bosonic part* are called *bosons*. The 1-summand of the superspace is called *fermionic part* and vectors belonging to *fermionic part* are called *fermions*. All endomorphisms of the superspace $C^{p|q}$ forms a Lie super-algebra, which is denoted by $gl(p|q)$. Concretely, the Lie super-algebra consists of $(p|q) \times (p|q)$ matrices $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. For this super-algebra, its bosonic part, or the even part, consists of $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$; the fermionic part, the odd part, consists of $\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$. The Lie super-bracket is defined as super-commutator $[X, Y] = XY - (-1)^{deg(X)deg(Y)}YX$, where $deg(X)$ is 0 if X is a boson and $deg(X)$ is 1 if X is a fermion. And the definition extends linearly to the whole $gl(1|1)$.

The Lie super-algebra $gl(1|1)$ is generated by the following four elements:

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad G = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

which satisfy the following relations:

$$\begin{aligned} [X, Y] &= XY - (-1)^{\deg X \deg Y} YX = E \\ X^2 &= 0, \qquad Y^2 = 0, \\ [G, X] &= X, \qquad [G, Y] = -Y \\ [E, G] &= [E, X] = [E, Y] = 0, \end{aligned}$$

where

$$\deg E = \deg G = 0,$$
$$\deg X = \deg Y = 1.$$

It is known those relations define the super-algebra $gl(1|1)$.

2.2 Quantization of superalgebra $gl(1|1)$

The universal enveloping algebra $Ugl(1|1)$ admits a deformation, which results in the quantum super-algebra $U_q gl(1|1)$ [9][10]. Notice that the multiplication in $Ugl(1|1)$ is still written as regular product. We deform one defining relation of $Ugl(1|1)$ about two odd generators X and Y. That is,

$$[X, Y] = XY - (-1)^{\deg X \deg Y} YX = \frac{q^E - q^{-E}}{q - q^{-1}}.$$

To make sense this non-algebraic expression, we join power series in E to the algebra. Denote the parameter of the deformation by h, put $q = e^h$ and write

$$H = q^{E/2} = e^{hE/2} = \sum_{n=0} \frac{h^n E^n}{2^n n!}.$$

Then, the quantum super-algebra $U_q gl(1|1)$ has a generator set $\{ G, X, Y, H, H^{-1} \}$ and the defining relations:

$$HX = XH, \quad HY = YH, \quad HG = GH, \quad H^{-1}Y = YH^{-1} \tag{1}$$
$$H^{-1}X = XH^{-1}, \quad H^{-1}G = GH^{-1}, \quad HH^{-1} = H^{-1}H = 1, \tag{2}$$
$$GX - XG = X, \quad GY - YG = -Y, \quad X^2 = 0, \quad Y^2 = 0, \tag{3}$$
$$XY + YX = \frac{H^2 - H^{-2}}{q - q^{-1}}. \tag{4}$$

If we denote the tensor algebra (the free associative algebra generated by G, X, Y, H, H^{-1} over the complex field C) by T, we have $U_q gl(1|1) = T/I$, where I is a two-sided ideal in T generated by the following elements:

$$\begin{aligned}
&HG - GH, \qquad H^{-1}G - GH^{-1}, \\
&HY - YH, \qquad H^{-1}Y - YH^{-1}, \\
&HX - XH, \qquad H^{-1}X - XH^{-1}, \\
&X^2, \quad Y^2, \quad HH^{-1} - 1, \quad H^{-1}H - 1, \\
&GX - XG - X, \qquad GY - YG + Y, \\
&XY + YX - \frac{H^2 - H^{-2}}{q - q^{-1}}.
\end{aligned}$$

In the quantum super-algebra $U_q gl(1|1)$, there is a structure of Hopf algebra when it is properly equipped with a comultiplication, a counit and an antipode.

The comultiplication as an algebraic map

$$\Delta : \qquad U_q gl(1|1) \to U_q gl(1|1) \otimes U_q gl(1|1)$$

can be defined as follows for generators,

$$\begin{aligned}
&\Delta(H) = H \otimes H, \quad \Delta(H^{-1}) = H^{-1} \otimes H^{-1}, \quad \Delta(G) = G \otimes 1 + 1 \otimes G, \\
&\Delta(X) = X \otimes H^{-1} + H \otimes X, \qquad \Delta(Y) = Y \otimes H^{-1} + H \otimes Y,
\end{aligned}$$

and algebraically be extended to the whole $U_q gl(1|1)$ with preserving grades. The graded tensor product follows

$$a \otimes b \cdot c \otimes d = (-1)^{\deg b \deg c} ac \otimes bd.$$

The counit as an algebraic map

$$\varepsilon : \qquad U_q gl(1|1) \to C$$

can be assigned for generators as

$$\varepsilon(H)=1, \qquad \varepsilon(H^{-1})=1, \qquad \varepsilon(1)=1,$$
$$\varepsilon(G)=0, \qquad \varepsilon(X)=0, \qquad \varepsilon(Y)=0,$$

and then be algebraically extended to the $U_q gl(1|1)$.

The antipode as an anti-algebraic map

$$S: \qquad U_q gl(1|1) \to U_q gl(1|1)$$

can be assigned for generators as

$$\begin{aligned} S(H) &= H^{-1}, \qquad S(H^{-1})=H, \\ S(G) &= -G, \qquad S(X)=-X, \qquad S(Y)=-Y \end{aligned}$$

and then be algebraically extended to the $U_q gl(1|1)$. Here the care need to be take for the degrees of elements in the following fashion

$$S(AB)=(-1)^{\deg A \deg B} S(B)S(A).$$

We can easily verify that Δ, ε and S are all well-defined as graded algebraic maps. So $(U_q gl(1|1), \cdot, \Delta, \eta, \varepsilon, S)$ is a Hopf super-algebra, which is called quantum supergroup, where η is the unit.

In order to study integrals of $U_q gl(1|1)$, linear basis is important. We therefore give a linear base in this section. Let's first give some useful relations among generators.

Lemma 2.1.

$$\begin{aligned} G^n X &= X(G+1)^n, \qquad XG^n=(G-1)^n X, \\ G^n Y &= Y(G-1)^n, \qquad YG^n=(G+1)^n X. \end{aligned}$$

The proof of this lemma is straightforward by using mathematical induction.

Proposition 2.1. *$\{H^l G^n X^\delta Y^\tau : l \in Z,\ n \in Z^+,\ \delta, \tau \in Z_2\} = \Lambda$ forms a basis of quantum supergroup $U_q gl(1|1)$, where Z^+ is the set of nonnegative integers.*

Proof. We can directly verify this proposition by using the lemma 2.1 and basic definitions. We also can verify this proposition by using standard arguments as in the proof of the classical PBW theorem. Since the idea I is not homogeneous, $T/I = U_q gl(1|1)$ is not graded. However, it is a filtered algebra. Therefore, the proposition is true [7][8]. □

3 Integrals of $U_q gl(1|1)$

Let H be a Hopf algebra with multiplication $\cdot$, unit η, comultiplication Δ, counit ϵ and antipode S. A right integral $\int^r$ on H is an element of H^*, the dual space of H, such that

$$(\int^r \otimes id) \circ \Delta = \eta \circ \int^r,$$

or

$$\int^r \cdot x^* = \pi(x^*) \int^r,$$

where $x^* \in H^*$ and π is an augmentation of H^* given by $\pi(x^*) = \langle x^*, 1_H \rangle$. Similarly, we can define left integrals on Hopf algebras [5][6]. For quantum supergroup $U_q gl(1|1)$, we will prove there is no non-zero right integral or non-zero left integral. We give a useful lemma about the comultiplication of G^n. It can be confirmed by induction.

Lemma 3.1. *For $n \geq 1$,*

$$\Delta(G^n) = \sum_{k=0}^{n} \binom{n}{k} G^{n-k} \otimes G^k.$$

Theorem 3.1. *There does not exist any non-zero right integral on the quantum supergroup $U_q gl(1|1)$.*

Proof. Suppose $\int^r \in U_q gl(1|1)^*$ is a non-zero right integral, according to the definition of right integrals, $(\int^r \otimes id)\Delta = \eta \circ \int^r$. So for $x \in U_q gl(1|1)$,

$$(\int^r \otimes id)\Delta(x) = \eta \circ \int^r (x),$$

$$\text{or} \qquad \sum_{(x)} \int^r (x_{(1)}) x_{(2)} = \int^r (x) 1.$$

There is a linear basis for quantum supergroup $U_q gl(1|1)$ as in Proposition 2.1. In order to prove the nonexistence of nonzero integrals, we only need to verify that the value of the integral on each basis element of Λ must be zero. For the convenience, we divide the basis elements into 4 types. The same type elements share some common properties, so that the verification becomes easier.

Type 1 basis elements : H^l, G^n, X, Y and the unit element 1. Apply the definition of right integrals to X, we have $(\int^r \otimes id)\Delta(X) = \int^r(X)1$. Then

$$\int^r (X) H^{-1} + \int^r (H) X = \int^r (X) 1.$$

Since 1, H^{-1}, and X are all basis elements, the coefficients of their linear combination must be zeroes. So, $\int^r(X) = 0$. Similarly, we can get $\int^r(Y) = 0$. When $l \neq 0$, $(\int^r \otimes id)\Delta(H^l) = \int^r(H^l)1$ implies that

$\int^r(H^l)H^l = \int^r(H^l)1$. That means $\int^r(H^l) = 0$ since H^l and 1 are not parallel vectors. When $n = 1$, $(\int^r \otimes id)\Delta(G) = \int^r(G)1$ implies that $\int^r 1 = 0$. When $n = 2$, $(\int^r \otimes id)\Delta(G^2) = \int^r(G^2)1$ implies that $\int^r G = 0$. Suppose $\int^r G^k = 0$ for $k \leq n$, then use Lemma 3.1, we have

$$\sum_{k=0}^{n+2} \binom{n+2}{k} \int^r (G^{n+2-k})G^k = \int^r (H^l G^{n+2})1.$$

This implies that $\int^r G^{n+1} = 0$. Therefore, for any positive integer n, $\int^r G^n = 0$.

Type 2 basis elements: H^lG^n, H^lX, H^lY, G^nX, G^nY and XY. Let's compute the integral at each of these elements. From $(\int^r \otimes id)\Delta(XY) = \int^r(XY)1$, we have

$$\int^r(XY)H^{-2} - \int^r(HY)H^{-1}X + \int^r(HX)H^{-1}Y + \int^r(H^2)XY = \int^r(XY)1.$$

Since this is a linear combination of 5 basis elements, $\int^r(XY)$ must be zero. For $l \neq 0$ and $n \neq 0$, from $(\int^r \otimes id)\Delta(H^lG^n) = \int^r(H^lG^n)1$, we have the equation

$$\sum_{k=0}^{n} \binom{n}{k} \int^r (H^lG^{n-k})H^lG^k = \int^r (H^lG^n)1.$$

The linear independence of basis elements implies that $\int^r(H^lG^n) = 0$. When $l \neq 1$, from $(\int^r \otimes id)\Delta(H^lX) = \int^r(H^lX)1$ we get $\int^r(H^lX) = 0$. When $l = 1$, we compute

$$\Delta(HGX) = HGX \otimes 1 + HX \otimes G + H^2G \otimes HX + H^2 \otimes HGX,$$

and apply integral, we get $\int^r(HX)G + \int^r(H^2G)HX$. Therefore $\int^r(HX) = 0$. Since $\Delta(G^nX) = \sum_{k=0}^{n} \binom{n}{k}(G^{n-k}X \otimes H^{-1}G^k + HG^{n-k} \otimes G^kX)$, then

$\sum_{k=0}^{n}\binom{n}{k}(\int^r(G^{n-k}X)H^{-1}G^k + \int^r(HG^{n-k})G^kX) = \int^r(G^nX)1$. This implies $\int^r(G^nX) = 0$. Since the generator Y plays exactly rule as X does in generating relations, we can get $\int^r(H^lY) = 0$ and $\int^r(G^nY) = 0$ without detailed computation.

Type 3 basis elements: H^lXY, G^nXY, H^lG^nX, H^lG^nY. When $l \neq 2$, we compute that $\Delta(H^lXY) = H^lXY \otimes H^{l-2} - H^{l+1}Y \otimes H^{l-1}X + H^{l+1}X \otimes H^{l-1}Y + H^{l+2} \otimes H^lXY$, and apply the definition of right integral. By the linear independence of basis elements, we must have $\int^r(H^lXY) = 0$. When $l = 2$, write $(\int^r \otimes id)\Delta(H^2GXY) = \int^r(H^2GXY)1$ out, and cancel the first term on both sides, we have

$$\begin{aligned}
&-\int^r(H^3GY)HX + \int^r(H^3GX)HY + \int^r(H^4G)H^2XY \\
&+\int^r(H^2XY)G - \int^r(H^3Y)HGX + \int^r(H^3X)HGY \\
&+\int^r(H^4)H^2GXY = 0.
\end{aligned}$$

This equation implies $\int^r(H^2XY) = 0$. When $l \neq 1$, by Lemma 3.1, we have $\Delta(H^lG^nX) = \sum_{k=0}^{n}\binom{n}{k}(H^lG^{n-k}X \otimes H^{l-1}G^k + H^{l+1}G^{n-k} \otimes H^lG^kX)$. The equation

$$\sum_{k=0}^{n}\binom{n}{k}(\int^r(H^lG^{n-k}X)H^{l-1}G^k + \int^r(H^{l+1}G^{n-k})H^lG^kX) = \int^r(H^lG^nX)1$$

implies that $\int^r(H^lG^nX) = 0$. When $l = 1$, we compute $(\int^r \otimes id)\Delta(HG^2X) =$

$\int^r (HG^2X)1$, and we have

$$\begin{aligned}&\int^r (HG^2X)1 + 2\int^r (HGX)G + \int^r (HX)G^2 + \\ &+ \int^r (H^2G^2)HX + 2\int^r (H^2G)HGX + \int^r (H^2)G^2X \\ &= \int^r (HG^2X)1.\end{aligned}$$

From this equation, we have $\int^r (HGX) = 0$. Suppose $\int^r (HG^kX) = 0$ for $k \le n$, we can check $\int^r (HG^{n+1}X) = 0$ by using Lemma 3.1. Write $(\int^r \otimes id)\Delta(HG^{n+2}X) = \int^r (HG^{n+2}X)1$ out as

$$\sum_{k=0}^{n+2} \binom{n+2}{k} (\int^r (HG^{n+2-k}X)G^k + \int^r (HG^{n+2-k})G^kX),$$

we see $\int^r (HG^{n+1}X) = 0$. We can get $\int^r (H^lG^nY) = 0$ by replace X by Y in above proof.

By using Lemma 3.1, we have

$$\begin{aligned}\Delta(G^nXY) = \sum_{k=0}^{n} \binom{n}{k} (&G^{n-k}XY \otimes H^{-2}G^k - HG^{n-k}Y \\ &\otimes H^{-1}G^kX + HG^{n-k}X \otimes H^{-1}G^kY + H^2G^{n-k} \otimes G^kXY).\end{aligned}$$

Apply the definition of right integrals, we get a linear combination of some basis elements. Therefore, each coefficient must be zero, so $\int^r (G^nXY) = 0$.

Type 4 basis elements: $H^lG^nXY = x$ where $l \neq 0$ and $n \neq 0$. We compute the comultiplication of x by using Lemma 3.1.

$$\Delta(H^lG^nXY) = H^l \otimes H^l \cdot (\sum_{k=0}^{n} \binom{n}{k} G^{n-k} \otimes G^k) \cdot$$
$$(X \otimes H^{-1} + H \otimes X) \cdot (Y \otimes H^{-1} + H \otimes Y)$$
$$= \sum_{k=0}^{n} \binom{n}{k} (H^lG^{n-k}XY \otimes H^{l-2}G^k - H^{l+1}G^{n-k}Y \otimes H^{l-1}G^kX$$
$$+H^{l+1}G^{n-k}X \otimes H^{l-1}G^kY + H^{l+2}G^{n-k} \otimes H^lG^kXY).$$

Using the definition of right integral, we have

$$\sum_{k=0}^{n} \binom{n}{k} (\int^r (H^lG^{n-k}XY)H^{l-2}G^k - \int^r (H^{l+1}G^{n-k}Y)H^{l-1}G^kX$$
$$+ \int^r (H^{l+1}G^{n-k}X)H^{l-1}G^kY + \int^r (H^{l+2}G^{n-k})H^lG^kXY)$$
$$= \int^r (H^lG^nXY)1$$

The left-hand side of this equation is a linear combination of some basis elements. When $l \neq 2$, the vector $\int^r (H^lG^nXY)1$ is presented by a linear combination of some other basis elements. But 1 is also a basis element. Therefore, each coefficient must be zero. So, $\int^r (H^lG^nXY) = 0$. When $l = 2$, for any positive integer n, the first term of the left-hand side of the above equation is the same as its right-hand side. After these two term are canceled out, the rest integral values must be zero since they are coefficients of a linear combination of basis vectors which is equal to zero. If we compute $\Delta(H^2G^2XY)$, and apply definition of right integrals, we get $\int^r (H^2GXY) = 0$. Now let's suppose $\int^r (H^2G^kXY) = 0$ for $k \leq n$, we then compute $\Delta(H^2G^{n+2}XY)$. After integration, we have $\int^r (H^2G^{n+1}XY) = 0$.

Considering all the cases, we can conclude that the only situation where $(\int^r \otimes 1)\Delta(x) = \eta \circ \int^r(x)$ is satisfied is $\int^r = 0$. Therefore, there does not exit non-zero right integral on $U_q gl(1|1)$. $\square$

Theorem 3.2. *There does not exit non-zero left integral on quantum supergroup $U_q gl(1|1)$.*

The proof of this theorem is similar to that of Theorem 3.1.

4 Hopf subalgebras of $U_q gl(1|1)$ and their integrals

From Theorems 3.1 and 3.2, it is impossible to use quantum supergroup $U_q gl(1|1)$ to construct Hennings type invariants of 3-manifolds. However,There are several Hopf subalgebras of the quantum supergroup $U_q gl(1|1)$. We may construct integrals for them.

The Hopf subalgebra of $U_q gl(1|1)$ generated by H, X and Y is denoted by $\langle H, X, Y\rangle$, the Hopf subalgebra generated by H is denoted by $\langle H\rangle$, and the Hopf subalgebra generated by G is denoted by $\langle G\rangle$. These three subalgebras are all infinite dimensional. The following theorem give their linear basis.

Theorem 4.1. *The Hopf subalgebra $\langle H, X, Y\rangle$ has a linear basis given by*

$$\{H^l X^\delta Y^\tau : l \in Z, \delta, \tau \in Z_2\}.$$

The Hopf subalgebra $\langle H\rangle$ has a linear basis H^l, where $l \in Z$. The Hopf subalgebra $\langle G\rangle$ has a linear basis G^n, where $l \in Z_{\geq 0}$.

The proof of this theorem is similar as that of Proposition 2.1.

Now let's construct integrals on $\langle H, X, Y\rangle$. We define a linear function $\int^r$,

$$\int^r \quad : \quad \langle H, X, Y\rangle \longrightarrow C$$

by assigning $\int^r H^2XY = 1$, $\int^r B = 0$ for any other basis element B, and then linearly extending to the whole algebra. We may write $\int^r = (H^2XY)^*$, which is an element of the dual space $\langle H, X, Y\rangle^*$.

Theorem 4.2. *$\int^r$ is a right integral on the Hopf algebra $\langle H, X, Y\rangle$. All right integrals form an one-dimensional subspace of $\langle H, X, Y\rangle^*$.*

Proof. We check if the functional $\int^r$ satisfies $(\int^r \otimes 1)\Delta(x) = \eta \circ \int^r(x)$ for any basis element x, then $\int^r$ must be $(H^2XY)^*$ or some scale k multiple of $(H^2XY)^*$. This implies the integral space is one dimensional. There are three types of basis elements. Type one are H^l, X and Y, where $l \neq 0$. Type two are H^lX, H^lY, XY and the unit 1. Type three is H^lXY, where $l \neq 0$. We can check the $\int^r$ at each basis element.

For type one basis elements, as the proof of Theorem 3.1, we can easily get that $\int^r(H^l) = 0$, $\int^r(X) = 0$, and $\int^r(Y) = 0$. For type two basis elements, as the proof of Theorem 3.1, we have $\int^r(H^lX) = 0$ and $\int^r(H^lY) = 0$ when $l \neq 1$. And $(\int^r \otimes 1)\Delta(XY) = \int^r(XY)1$ just implies that $\int^r(XY) = 0$, $\int^r(HX) = 0$ and $\int^r(HY) = 0$. For type three basis elements, when $l \neq 2$, as the proof of Theorem 3.1, we have $\int^r(H^lXY) = 0$. But, when $l = 2$,

$$
\begin{aligned}
&(\int^r \otimes 1)\Delta(H^2XY) \\
&= \int^r(H^2XY)1 - \int^r(H^3Y)HX + \int^r(H^3X)HY + \int^r(H^4)H^2XY \\
&= \int^r(H^2XY)1.
\end{aligned}
$$

This just implies we can assign non-zero number to $\int^r(H^2XY)$, and they form a subspace with dimension one. $\square$

We define another linear functional $\int^l$ as

$$\int^l \quad : \quad \langle H, X, Y\rangle \longrightarrow C$$

For basis element x, $\int^l x = 1$ when $x = H^{-2}XY$, and $\int^l x = 0$ if $x \neq H^{-2}XY$. This functional is the dual of $H^{-2}XY$, $\int^l = (H^{-2}XY)^*$.

Theorem 4.3. *$\int^l$ is a left integral on Hopf subalgebra $\langle H, X, Y\rangle$. All left integrals form an one-dimensional subspace of $\langle H, X, Y\rangle^*$.*

The proof is similar as that of Theorem 4.3.

From the proof of Theorem 3.1, we have the following theorem about integrals of Hopf subalgebras $\langle H\rangle$ and $\langle G\rangle$.

Theorem 4.4. *The Hopf subalgebra $\langle H\rangle$ is the center of the Hopf algebras $U_qgl(1|1)$, and it has a left integral which is also a right integral defined by the dual of the unit element, $\int 1 = 1$ and $\int H^l = 0$. The Hopf subalgebra $\langle G\rangle$ does not have a right or left integral.*

5 The quantum superalgebra $U_q gl(n|m)$ and their sub superalgebras

The quantum superalgebra $U_q gl(n|m)$ is a free associative algebra over $\mathbb{C}$ with a parameter $q \in \mathbb{C}$ generated by generators k_i, k_i^{-1}, where $i = 1, 2, \cdots, n+m$, and generators e_j, f_j, where $j = 1, 2, \cdots, n+m-1$. The defining relations, the Cartan-Kac relations, e-Serre relations, and f-Serre relations, are given in the following [12].

The Cartan-Kac relations:

$$
\begin{aligned}
& k_i k_j = k_j k_i, \qquad k_i k_i^{-1} = k_i^{-1} k_i = 1; \\
& k_i e_j k_i^{-1} = q^{(\delta_{ij} - \delta_{ij+1})/2} e_j, \qquad k_i f_j k_i^{-1} = q^{-(\delta_{ij} - \delta_{ij+1})/2} f_j; \\
& e_i f_j - f_j e_i = 0, \quad \text{if} \quad i \neq j; \\
& e_i f_i - f_i e_i = (k_i^2 k_{i+1}^{-2} - k_{i+1}^2 k_i^{-2})/(q - q^{-1}), \quad \text{if} \quad i \neq n; \\
& e_n f_n + f_n e_n = (k_n^2 k_{n+1}^2 - k_n^{-2} k_{n+1}^{-2})/(q - q^{-1}).
\end{aligned}
$$

The Serre relations for the e_i (e-Serre relations):

$$
\begin{aligned}
& e_i e_j = e_j e_i \quad \text{if} \quad |i-j| \neq 1, \quad e_n = 0; \\
& e_i^2 e_{i+1} - (q + q^{-1}) e_i e_{i+1} e_i + e_{i+1} e_i^2 = 0, \quad i \neq n, n+m-1; \\
& e_n e_{n-1} e_n e_{n+1} + e_{n-1} e_n e_{n+1} e_n + e_n e_{n+1} e_n e_{n-1} + \\
& e_{n+1} e_n e_{n-1} e_n - (q + q^{-1}) e_n e_{n-1} e_{n+1} e_n = 0.
\end{aligned}
$$

The Serre relations for the f_i (f-Serre relations): the relations are obtained by replacing every e_i by f_i in e-Serre relations above.

The $\mathbb{Z}_2$-grading is defined by the requirement that the only odd generators are e_n and f_n. It can shown that $U_q gl(n|m)$ is a Hopf superalgebra with comultiplication Δ, counit ε and antipode S defined for generators as follows and then graded-algebraically extended to the whole algebra.

$$\begin{aligned}
\varepsilon(e_j) &= \varepsilon(f_j) = 0, \quad \varepsilon(k_i) = 1. \\
\Delta(k_i) &= k_i \otimes k_i, \\
\Delta(e_j) &= e_j \otimes k_j k_{j+1}^{-1} + k_j^{-1} k_{j+1} \otimes e_j, \quad \text{if} \quad j \neq n, \\
\Delta(e_n) &= e_n \otimes k_n k_{n+1} + k_n^{-1} k_{n+1}^{-1} \otimes e_n, \\
\Delta(f_j) &= f_j \otimes k_j k_{j+1}^{-1} + k_j^{-1} k_{j+1} \otimes f_j, \quad \text{if} \quad j \neq n, \\
\Delta(f_n) &= f_n \otimes k_n k_{n+1} + k_n^{-1} k_{n+1}^{-1} \otimes f_n. \\
S(k_i) &= k_i^{-1}, \\
S(e_j) &= -q e_j, \quad S(f_j) = -q f_j, \quad \text{if} \quad i \neq n, \\
S(e_n) &= -e_n, \quad S(f_n) = -f_n.
\end{aligned}$$

Lemma 5.1. *The sub super algebra A_q of $U_q gl(n|m)$ generated by $k_n^{\pm 1}$, $k_{n+1}^{\pm 1}$, e_n and f_n is isomorphic to $U_q gl(1|1)$. The A_q is only subalgebra of $U_q gl(n|m)$ which is isomorphic to $U_q gl(1|1)$.*

Proof. Denote the sub super algebra generated by these 6 generators by A_q. Then A_q has the following defining relations:

$$k_n k_n^{-1} = k_n^{-1} k_n = 1, \quad k_{n+1} k_{n+1}^{-1} = k_{n+1}^{-1} k_{n+1} = 1; \tag{5}$$

$$k_n k_{n+1} = k_{n+1} k_n; \quad k_n e_n k_n^{-1} = q^{\frac{1}{2}} e_n, \quad k_n f_n k_n^{-1} = q^{-\frac{1}{2}} f_n; \tag{6}$$

$$k_{n+1} e_n k_{n+1}^{-1} = q^{-\frac{1}{2}} e_n, \quad k_{n+1} f_n k_{n+1}^{-1} = q^{\frac{1}{2}} f_n; \quad e_n^2 = 0, \tag{7}$$

$$e_n f_n + f_n e_n = (k_n^2 k_{n+1}^2 - k_n^{-2} k_{n+1}^{-2})/(q - q^{-1}); \quad f_n^2 = 0. \tag{8}$$

We know $U_q gl(1|1)$ has 5 generators and the defining relations (1)-(4). Let E_{ij} be a 2×2 matrix whose (i, j) entry is 1 and the rest entries are all zero. We define a map $\rho : \quad A_q \longrightarrow U_q gl(1|1)$ by assigning an element in $U_q gl(1|1)$ to each generator of A_q as follows:

$$
\begin{aligned}
&\rho(k_n) = q^{E_{11}/2}, \qquad \rho(k_n^{-1}) = q^{-E_{11}/2}, \\
&\rho(k_{n+1}) = q^{E_{22}/2} = q^{G/2}, \qquad \rho(k_{n+1}^{-1}) = q^{-E_{22}/2} = q^{-G/2}, \\
&\rho(e_n) = E_{12} = Y, \qquad \rho(f_n) = E_{21} = X,
\end{aligned}
$$

then graded-algebraically extend to the whole algebra A_q. ρ will be Hopf algebra isomorphism preserving grading of elements. We first verify ρ is an $1-1$ and onto algebraic isomorphism. It is not hard to get defining relations for $U_q gl(1|1)$ from the defining relations (5)-(8) of A_q by using the map ρ. The generators H and H^{-1} of $U_q gl(1|1)$ are given by $\rho(k_n k_{n+1}) = H$ and $\rho(k_n^{-1} k_{n+1}^{-1}) = H^{-1}$. The generator G is given by the second term of the Taylor expansion $\rho(k_{n+1}) = q^{G/2}$. To see that, set $q = e^{2h}$, then $q^{G/2} = e^{hG} = I + hG + \frac{1}{2!} h^2 G^2 + \cdots$. The relations $HH^{-1} = H^{-1} H = 1$ can be obtained by writing $\rho(k_n k_n^{-1} k_{n+1} k_{n+1}^{-1}) = 1$ in two ways.

$$
\begin{aligned}
HX &= \rho(k_n k_{n+1}) \rho(f_n) = \rho(k_n k_{n+1} f_n) = \rho(k_n q^{1/2} f_n k_{n+1}) \\
&= \rho(q^{1/2} k_n f_n k_{n+1}) = \rho(q^{1/2} q^{-1/2} f_n k_n k_{n+1}) \\
&= \rho(f_n) \rho(k_n k_{n+1}) = HX.
\end{aligned}
$$

Similarly, we can get $H^{-1} X = X H^{-1}$, $HY = YH$, and $H^{-1} Y = Y H^{-1}$. To get $HG = GH$ and $H^{-1} G = G H^{-1}$, expand $H q^{G/2} = \rho(k_n k_{n+1} k_{n+1}) = \rho(k_{n+1} k_n k_{n+1}) = q^{G/2} H$, and compare both sides in terms of parameter h. S-ince $\rho(k_{n+1} f_n) = \rho(q^{1/2} f_n k_{n+1})$, then $q^{G/2} X = q^{1/2} X q^{G/2}$. Using substitution

$q = e^{2h}$, we have $e^{hG}X = e^h X q^{hG}$. Expand both sides,

$$\begin{aligned}
&X + hGX + \frac{1}{2!}h^2G^2X + \frac{1}{3!}h^3G^3X + \cdots \\
&= X + hXG + \frac{1}{2!}h^2XG^2 + \frac{1}{3!}h^3XG^3 + \cdots + hX + h^2XG + \frac{1}{2!}h^3XG^2 + \\
&\frac{1}{3!}h^4XG^3 + \cdots + \frac{1}{2!}h^2X + \frac{1}{2!}h^3XG + \cdots,
\end{aligned}$$

we have $GX = XG + X$, $G^2X = X(G+1)^2$, and $G^lX = X(G+1)^l$ for any positive integer l. From Lemma 2.1, only essential relation is $GX = XG+X$. Similarly, we can get $GY = YG-Y$. From $\rho(f_ne_n+e_nf_n) = \rho(\frac{k_n^2k_{n+1}^2-k_n^{-2}k_{n+1}^{-2}}{q-q^{-1}})$, we have $XY+YX = \frac{H^2-H^{-2}}{q-q^{-1}}$. The $e_n^2 = 0$ and $f_n^2 = 0$ give $X^2 = 0$ and $Y^2 = 0$. So, the image of the generators of A_q generate $U_qgl(1|1)$. ρ is an $1-1$ and onto algebraic map.

Now we need to verify ρ is a Hopf algebra map. If use subscript A for structure maps in A_q, then we need check $\varepsilon_A = \varepsilon\rho$, $\Delta\rho = (\rho\otimes\rho)\Delta_A$ and $\rho S_A = S\rho$. Or, use these equations to define Hopf algebra structure on $U_qgl(1|1)$ from A_q. For example, $\varepsilon_A(k_{n+1}) = \varepsilon\rho(k_{n+1})$ gives $\varepsilon(G) = 0$. $\Delta\rho(k_{n+1}) = (\rho\otimes\rho)\Delta_A(k_{n+1})$ gives $\Delta(q^{G/2}) = q^{G/2}\otimes q^{G/2}$. Expand in terms of parameter h, and compare both sides, we have $\Delta(G) = G\otimes 1 + 1\otimes G$, $\Delta(G^2) = G^2\otimes 1 + 2G\otimes G + 1\otimes G^2$, $\Delta(G^l) = \sum_{k=0}^{l}\binom{l}{k}G^{l-k}\otimes G^k$ for any positive integer l. By Lemma 3.1, the only essential relation is $\Delta(G) = G\otimes 1 + 1\otimes G$.

$$\begin{aligned}
&\Delta\rho(f_n) = \Delta(X) \\
&= (\rho\otimes\rho)\Delta_A(f_n) = (\rho\otimes\rho)(f_n\otimes k_nk_{n+1} + k_n^{-1}k_{n+1}^{-1}\otimes f_n) \\
&= X\otimes H + H^{-1}\otimes X.
\end{aligned}$$

From $\rho S_A(k_{n+1}) = S\rho(k_{n+1})$ we have $S(q^{G/2}) = q^{-G/2}$. Expand it, we get $S(G) = -G$. Similarly, we can confirm all other defining relations for Hopf algebra structure of $U_q gl(1|1)$.

By the defining relations (5)-(8) of $U_q gl(n|m)$, the only odd generators are e_n and f_n. Therefore we can conclude that the A_q is only sub super algebra of $U_q gl(n|m)$ which is isomorphic to $U_q gl(1|1)$. $\square$

Theorem 5.1. *(this not right) Let H be an infinite-dimensional Hopf algebra over a field K. If H has a non-zero left (right) integral, then any infinite-dimensional Hopf subalgebra of H also has a non-zero left (right) integral.*

Proof. It is know that H has a non-zero left integral if and only if H contains a proper left coideal of finite codimension [13]. Suppose H has a non-zero left integral, then H has a proper left coideal of finite codimension. Denote this left coideal by C, then H/C is a finite dimensional space, and $\Delta(C) \subset H \otimes C$. Let H_1 be any infinite-dimensional Hopf subalgebra of H. Denote $H_1 \cap C$ by C_1. Then $\Delta(C_1) \subset H_1 \otimes C_1$, since $\Delta(H_1) \subset H_1 \otimes H_1$. That is, C_1 is a left coideal of H_1.

$\square$

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